THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010G University Mathematics 2014-2015
Suggested Solution to Test 1

1. (a) $\lim_{n\to\infty} \frac{5n^2-1}{n^2-3n+2} = \lim_{n\to\infty} \frac{5-\frac{1}{n^2}}{1-\frac{3}{n}+\frac{2}{n^2}} = 5$

   (b) $\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{n+2} = \lim_{n\to\infty} \left[\left(1+\frac{1}{2n}\right)^{2n}\right]^{1/2} \cdot (1+\frac{1}{2n})^2 = e^{1/2}\cdot 1 = e^{1/2}$

2. Note that $\frac{1}{\sqrt[3]{n^3+n}} \le \frac{1}{\sqrt[3]{n^3+i}} \le \frac{1}{\sqrt[3]{n^3}} = \frac{1}{n}$ for all $1\le i\le n$, so we have

$$\frac{1}{\sqrt[3]{n^3+n}}\cdot n \le \frac{1}{\sqrt[3]{n^3+1}} + \frac{1}{\sqrt[3]{n^3+2}} + \cdots + \frac{1}{\sqrt[3]{n^3+n}} \le \frac{1}{n}\cdot n = 1$$

   Note that $\lim_{n\to\infty} \frac{n}{\sqrt[3]{n^3+n}} = 1$. By sandwich theorem,

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^3+1}} + \frac{1}{\sqrt[3]{n^3+2}} + \cdots + \frac{1}{\sqrt[3]{n^3+n}} = 1.$$

3. (a) $\lim_{x\to 0} \frac{\tan 3x}{x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{3}{\cos 3x} = 1\cdot 3 = 3$

   (b) $\lim_{x\to +\infty} \frac{e^{x+1}+e^{-(x+1)}}{e^{x-1}-e^{-(x+1)}} = \lim_{x\to +\infty} \frac{e+e^{-2x-1}}{e^{-1}-e^{-2x-1}} = e^2$

   (c) $\lim_{x\to -\infty} \frac{x}{\sqrt{4x^2-x+1}} = \lim_{x\to -\infty} \frac{1}{\frac{1}{x}\sqrt{4x^2-x+1}} = \lim_{x\to -\infty} \frac{1}{-\sqrt{4-\frac{1}{x}+\frac{1}{x^2}}} = -\frac{1}{2}$

4. (a) $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = 0$ and $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0$. Therefore, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists and equals to 0, that means f is differentiable at $x=0$ and $f'(0)=0$.

   (b) If $x>0$, $f'(x)=2x$. If $x<0$, $f'(x)=0$. Combine them with the result in (a), we have

$$f'(x) = \begin{cases} 2x & \text{if} \quad x\ge 0 \\ 0 & \text{if} \quad x<0 \end{cases}$$

   We have $\lim_{h\to 0^+} \frac{f'(0+h)-f'(0)}{h} = \lim_{h\to 0^+} \frac{2h}{h} = 2$, but $\lim_{h\to 0^-} \frac{f'(0+h)-f'(0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0$. Therefore, $\lim_{h\to 0} \frac{f'(0+h)-f'(0)}{h}$ does not exists, i.e. f' is not differentiable at $x=0$.

5. Let $f(x)=x^n$, so f is differentiable everywhere.
   If $x>y>0$, by Mean Value Theorem, there exists $c\in(y,x)$ such that

$$\begin{aligned} \frac{f(x)-f(y)}{x-y} &= f'(c) \\ x^n-y^n &= nc^{n-1}(x-y) \end{aligned}$$

Note that $x > c > y$, so $x^{n-1} \geq c^{n-1} \geq y^{n-1}$, and we have

$$ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y).$$

6. (a) i. Put $x = y = 0$, we have $f(0) = [f(0)]^2$, so $f(0) = 0$ or 1. If we put 0 to the inequality in the second condition, we have $1 \leq f(0)$, so $f(0) = 1$.

ii. If $x > 0$, by the inequality in the second condition, we have

$$f(x) \geq 1 + x > 1.$$

iii. If $x < 0$, then

$$1 = f(0) = f(x - x) = f(x)f(-x).$$

Therefore, $f(x) = \dfrac{1}{f(-x)} > 0$.

If $a > b$, then

$$\begin{aligned} f(a) - f(b) &= f(b + (a-b)) - f(b) \\ &= f(b)f(a-b) - f(b) \\ &= f(b)(f(a-b) - 1) \\ &> 0 \end{aligned}$$

Note: $a - b > 0$, so $f(a-b) > 1$.

(b) We put h to the inequality in the second condition, we have

$$1 + h \leq f(h) \leq 1 + hf(h).$$

Also, $f(h) \leq 1 + hf(h)$ implies that $f(h) \leq \dfrac{1}{1-h}$ if $h < 1$.
Therefore, when $h < 1$,

$$1 + h \leq f(h) \leq \frac{1}{1-h}.$$

Note that $\lim_{h\to 0} 1 + h = \lim_{h\to 0} \dfrac{1}{1-h} = 1$, so by sandwich theorem, we have

$$\lim_{h\to 0} f(h) = 1 = f(0),$$

which implies f is continuous at $x = 0$.

(c) From the inequality in the second condition, we have

$$\begin{aligned} 1 + h \leq &\quad f(h) &\leq 1 + hf(h) \\ h \leq &\quad f(h) - 1 &\leq hf(h) \end{aligned}$$

If $h > 0$, we have $1 \leq \dfrac{f(h)-1}{h} \leq f(h)$ so by sandwich theorem, we have

$$\lim_{h\to 0^+} \frac{f(h)-1}{h} = 1.$$

Similarly, if $h < 0$, we have $1 \geq \dfrac{f(h)-1}{h} \geq f(h)$ so by sandwich theorem, we have

$$\lim_{h\to 0^-} \frac{f(h)-1}{h} = 1.$$

Therefore, $\lim_{h \to 0} \frac{f(h) - 1}{h} = 1$.

Now, we have

$$\begin{aligned} \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0} f(0) \cdot \frac{f(h) - 1}{h} \\ &= 1 \end{aligned}$$

which implies that f is differentiable at $x = 0$ and $f'(0) = 1$.